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Correlation functions of the *SU*(2)-invariant Thirring model with a boundary

Bo-Yu Hou†, Kang-Jie Shi†, Yan-Shen Wang‡ and Wen-Li Yang†

† Institute of Modern Physics, Northwest University, Xian 710069, People's Republic of China
‡ Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, People's Republic of China

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Abstract. Using the bosonization and oscillator expressions of bulk and boundary Zamolodchikov–Faddeev algebras, the correlators of Jost functions via Z' operators are obtained in the case of $SU(2)$-invariant Thirring model with a boundary.

1. Introduction

Correlation functions or form factors for many arbitrary local operators are very important to integrable theory. Recently, much progress has been made. In exact solvable lattice models [1–6], using the free boson realization of q-deformed affine algebras [7], types I and II vertex operators [8–11] which satisfy the Zamolodchikov–Faddeev (ZF) algebra have been provided and the correlation functions for the six-vertex model or XXZ-chain have been successfully calculated. In massive integrable quantum field theories, e.g. $SU(2)$-invariant Thirring ($SU(2)ITM$) and Sine–Gordon models (SGM), Lukyanov [12, 13] constructed two kinds of generator for the ZF algebra, local and asymptotic operators which behave quite similarly to type I and II operators. Lukyanov utilizing q-deformed free bosons and q-oscillators [14, 15] calculated the form factors for these models in an alternative way to that used by Smirnov [16, 17]. On the other hand, Ghoshal and Zamolodchikov [18] have proposed theories of integrable boundaries and boundary operators. The q-boson realization for the boundary operators of the XXZ-model and the corresponding multipoint correlation functions were obtained by Jimbo *et al* [19, 20]. In the present paper, we will continue our early work [21] in which the q-deformed oscillator realization of the boundary operators of $SU(2)ITM$ with a boundary was obtained. First, we calculate the correlators of the Jost functions via the $Z'_a(\alpha)$ operators expressed with q-oscillators, which are similar to those of the XXZ-model. Then, we let $q^{\frac{1}{4}}(= -e^{\epsilon\pi/2}) \to -1$ that means taking continuum limit [1], and obtain the correlators of the Jost functions of $SU(2)ITM$ with a boundary.

2. *SU*(2)-invariant Thirring model in the bulk

In this section, let us review Lukyanov's q-deformed oscillator realization of the ZF algebra [12, 13] for $SU(2)ITM$ (bulk theory) in the space π_z, which may be considered as a space of angular quantization of a massive integral model. Lukyanov provided two kinds of generator, $Z_a(\beta)$ which behaves like an asymptotic operator and $Z'_a(\alpha)$ which is related to local operators. (Here and after we shall adopt the convention that operators without a prime denote asymptotic operators and those with a prime denote local operators.) These generators have different normalizations:

$$\mathrm{i}Z_a(\beta_2)Z_b(\beta_1) = \frac{C_{ab}}{\beta_2 - \beta_1 - \mathrm{i}\pi} + \cdots \tag{1}$$

$$C_{ab}Z'_a(\alpha + \mathrm{i}\pi)Z'_b(\alpha) = \mathrm{i}$$
$$Z'_a(\alpha - \mathrm{i}\pi)Z'_b(\alpha) = \mathrm{i}C_{ab} \tag{2}$$

here and below summation over repeated indices is understood. $C_{ab} = (\sigma_1)_{ab}$ (the Pauli matrix) is the element of the charge conjugation matrix. α, $\beta \in C$ are the rapidities of the particles. These generators satisfy the ZF algebra,

$$Z_a(\beta_1)Z_b(\beta_2) = S^{cd}_{ab}(\beta_1 - \beta_2)Z_d(\beta_2)Z_c(\beta_1) \tag{3}$$
$$Z'_a(\alpha_1)Z'_b(\alpha_2) = S'^{cd}_{ab}(\alpha_1 - \alpha_2)Z'_d(\alpha_2)Z'_c(\alpha_1) \tag{4}$$

where the S-matrix $S^{cd}_{ab}(\beta)$ (or $S'^{cd}_{ab}(\alpha)$) satisfies the well known Yang–Baxter equation

$$S^{c_1c_2}_{a_1a_2}(\beta_1)S^{b_1c_3}_{c_1a_3}(\beta_1+\beta_2)S^{b_2b_3}_{c_2c_3}(\beta_2) = S^{c_2c_3}_{a_2a_3}(\beta_2)S^{c_1b_3}_{a_1c_3}(\beta_1+\beta_2)S^{b_1b_2}_{c_1c_2}(\beta_1) \tag{5}$$

the unitarity condition

$$S^{c_1c_2}_{a_1a_2}(\beta)S^{b_2b_1}_{c_2c_1}(-\beta) = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2} \tag{6}$$

and the crossing symmetry

$$S^{b_1b_2}_{a_1a_2}(\beta) = C_{a_1a_3}C_{b_1b_3}S^{b_2a_3}_{a_2b_3}(\mathrm{i}\pi - \beta). \tag{7}$$

Explicitly, the S-matrix for the $SU(2)$ Thirring model reads:

$$(S^{cd}_{ab}(\beta)) = (-S'^{cd}_{ab}(-\beta)) = \begin{pmatrix} 1 & & & \\ & \frac{\beta}{\mathrm{i}\pi-\beta} & \frac{\mathrm{i}\pi}{\mathrm{i}\pi-\beta} & \\ & \frac{\mathrm{i}\pi}{\mathrm{i}\pi-\beta} & \frac{\beta}{\mathrm{i}\pi-\beta} & \\ & & & 1 \end{pmatrix} S(\beta). \tag{8}$$

$S(\beta)$ can be Riemann–Hilbert decomposed into $g(\beta)$; $g(\beta)$ is required to be analytic in the lower half-plane $\mathrm{Im}\,\beta \leqslant 0$,

$$S(\beta) = \frac{g(-\beta)}{g(\beta)}$$
$$g(\beta) = k^{\frac{1}{2}}\frac{\Gamma\left(\frac{\mathrm{i}\pi-\beta}{2\mathrm{i}\pi}\right)}{\Gamma\left(\frac{-\beta}{2\mathrm{i}\pi}\right)} \tag{9}$$

where k is a constant. The ZF algebras (3) can be realized through the bosonic field $\phi(\beta)$ with the commutation relation

$$[\phi(\beta_1), \phi(\beta_2)] = \ln S(\beta_2 - \beta_1) \tag{10}$$

as follows

$$Z_+(\beta) = V(\beta)$$
$$Z_-(\beta) = \mathrm{i}[\chi V(\beta) + V(\beta)\chi]. \tag{11}$$

$V(\beta)$, χ are defined through the bosonic field $\phi(\beta)$:

$$
\begin{aligned}
V(\beta) \quad &\equiv \exp(\mathrm{i}\phi(\beta)) = \rho : \exp(\mathrm{i}\phi(\beta)) : \\
\langle u|\chi|v\rangle &\equiv \eta^{-1}\langle u| \int_c \frac{\mathrm{d}\gamma}{2\pi} \bar{V}(\gamma)|v\rangle \\
\bar{V}(\gamma) \quad &\equiv \exp(-\mathrm{i}\bar{\phi}(\gamma)) = \bar{\rho} : \exp(-\mathrm{i}\bar{\phi}(\gamma)) : \\
\bar{\phi}(\gamma) \quad &\equiv \phi\left(\gamma + \frac{\mathrm{i}\pi}{2}\right) + \phi\left(\gamma - \frac{\mathrm{i}\pi}{2}\right)
\end{aligned}
\tag{12}
$$

where ρ, $\bar{\rho}$ and η are constants, and dots implies the normal ordering of exponents. $\langle u|$ and $|v\rangle$ are some states in the (dual) Fock space of the bosonic field $\phi(\beta)$, and the integration contour C goes from $\mathrm{Re}\,\gamma = -\infty$ to $\mathrm{Re}\,\gamma = +\infty$ and lies above all singularities whose position depend on the vector $|v\rangle$ but below singularities depending on $|u\rangle$. To specify the Fock space F on which the local operators act, one needs the unique $SU(2)$ invariant vacuum state $|0\rangle$ under which the two-point function of the boson fields $\phi(\beta)$ reads

$$
\langle 0|\phi(\beta_1)\phi(\beta_2)|0\rangle = -\ln g(\beta_2 - \beta_1). \tag{13}
$$

It is easy to obtain the following two-point functions from (12, 13)

$$
\begin{aligned}
\langle 0|\bar{\phi}(\beta_1)\phi(\beta_2)|0\rangle &= \ln w(\beta_2 - \beta_1) \\
\langle 0|\bar{\phi}(\beta_1)\bar{\phi}(\beta_2)|0\rangle &= -\ln \bar{g}(\beta_2 - \beta_1)
\end{aligned}
\tag{14}
$$

where

$$
\begin{aligned}
w(\beta) &= \left[g\left(\beta + \frac{\mathrm{i}\pi}{2}\right) g\left(\beta - \frac{\mathrm{i}\pi}{2}\right)\right]^{-1} = k^{-1} \frac{2\pi}{\mathrm{i}\left(\beta + \frac{\mathrm{i}\pi}{2}\right)} \\
\bar{g}(\beta) &= \left[w\left(\beta + \frac{\mathrm{i}\pi}{2}\right) w\left(\beta - \frac{\mathrm{i}\pi}{2}\right)\right]^{-1} = -k^2 \frac{\beta(\beta + \mathrm{i}\pi)}{4\pi^2}.
\end{aligned}
\tag{15}
$$

Similar to (11), Lukyanov gave the following boson realization of local operators:

$$
\begin{aligned}
Z'_+(\alpha) &= V'(\alpha) \\
Z'_-(\alpha) &= \mathrm{i}[\chi' V'(\alpha) + V'(\alpha)\chi']
\end{aligned}
\tag{16}
$$

where

$$
\begin{aligned}
V'(\alpha) &\equiv \exp(\mathrm{i}\phi'(\alpha)) = \rho' : \exp(\mathrm{i}\phi'(\alpha)) : \\
\langle u|\chi'|v\rangle &\equiv \eta'^{-1}\langle u| \int_c \frac{\mathrm{d}\delta}{2\pi} \bar{V}'(\delta)|v\rangle \\
\bar{V}'(\delta) &\equiv \exp(-\mathrm{i}\bar{\phi}'(\delta)) = \bar{\rho}' : \exp(-\mathrm{i}\bar{\phi}'(\delta)) : \\
\bar{\phi}'(\delta) &\equiv \phi'\left(\delta + \frac{\mathrm{i}\pi}{2}\right) + \phi'\left(\delta - \frac{\mathrm{i}\pi}{2}\right).
\end{aligned}
\tag{17}
$$

The operators $Z_a(\beta)$, $Z'_a(\alpha)$ satisfy the commutation relation,

$$
Z_a(\beta) Z'_b(\alpha) = ab \tan\left(\frac{\pi}{4} + \mathrm{i}\frac{\beta - \alpha}{2}\right) Z'_b(\alpha) Z_a(\beta). \tag{18}
$$

The bosonic field $\phi(\beta)$ and $\phi'(\alpha)$ essentially carry singularities. Therefore, it cannot be expanded into an infinite sum of oscillator modes as the usual free field does. However, we can perform an oscillator realization by the ultraviolet regularization introduced by Lukyanov. We consider the field $\phi_\epsilon(\beta)$ and $\phi'_\epsilon(\alpha)$ which are defined on the finite interval

$$
-\frac{\pi}{\epsilon} \leqslant \beta (\text{or } \alpha) \leqslant \frac{\pi}{\epsilon}. \tag{19}
$$

$\phi(\beta)$ and $\phi'(\alpha)$ can be viewed as the proper limit of $\phi_\epsilon(\beta)$ and $\phi'_\epsilon(\alpha)$ as $\epsilon \to 0$ respectively. Bosonic field $\phi_\epsilon(\beta)$ and $\phi'_\epsilon(\alpha)$ may be expressed with the q-oscillators as follows

$$\phi_\epsilon(\beta) = \frac{1}{\sqrt{2}}(Q - \epsilon\beta P) + \sum_{m=1}^{+\infty} \frac{a_m}{\mathrm{i}\sinh(m\epsilon\pi)} \exp(\mathrm{i}m \in \beta) - \sum_{m=1}^{+\infty} \frac{a_{-m}}{\mathrm{i}\sinh(m\epsilon\pi)} \exp(-\mathrm{i}m\epsilon\beta)$$
$$\phi'_\epsilon(\alpha) = -\frac{1}{\sqrt{2}}(Q - \epsilon\alpha P) - \sum_{m=1}^{+\infty} \frac{a'_m}{\mathrm{i}\sinh(m\epsilon\pi)} \exp(\mathrm{i}m\epsilon\alpha) + \sum_{m=1}^{+\infty} \frac{a'_{-m}}{\mathrm{i}\sinh(m\epsilon\pi)} \exp(-\mathrm{i}m\epsilon\alpha) \tag{20}$$

where q-oscillators satisfy

$$\begin{aligned}
&[P, Q] = -\mathrm{i} \\
&[a_m, a_n] = \frac{1}{m}\sinh\left(\frac{m\epsilon\pi}{2}\right)\sinh(m\epsilon\pi)\exp\left(\frac{|m|\epsilon\pi}{2}\right)\delta_{m+n,0} \\
&[a'_m, a'_n] = \frac{1}{m}\sinh\left(\frac{m\epsilon\pi}{2}\right)\sinh(m\epsilon\pi)\exp\left(-\frac{|m|\epsilon\pi}{2}\right)\delta_{m+n,0} \\
&a'_m = \exp\left(-\frac{|m|\pi\epsilon}{2}\right)a_m.
\end{aligned} \tag{21}$$

Here the Fock space F_ϵ of the q-oscillator is defined via the Fock vacuum $|p\rangle$, which are not physics vacuum states in the π_z representation. The q-oscillators act on these Fock vacua as follows [8]:

$$a_m|p\rangle = 0, (m > 0) \qquad P|p\rangle = p|p\rangle.$$

Note that the action of e^Q on the state $|p\rangle$ shifts the eigenvalue of P by 1 and does not affect the actions of $a_{\pm m}$. In analogy to (11, 16), we have

$$\begin{aligned}
Z_{\epsilon+}(\beta) &= \exp\left(\frac{\mathrm{i}}{4}\epsilon\beta\right)V_\epsilon(\beta) \\
Z_{\epsilon-}(\beta) &= -\mathrm{i}\exp\left(-\frac{\mathrm{i}}{4}\epsilon\beta\right)\left[\exp\left(\frac{\epsilon\pi}{4}\right)\chi_\epsilon V_\epsilon(\beta) + \exp\left(-\frac{\epsilon\pi}{4}\right)V_\epsilon(\beta)\chi_\epsilon\right] \\
Z'_{\epsilon+}(\alpha) &= \exp\left(\frac{\mathrm{i}}{4}\epsilon\alpha\right)V'_\epsilon(\alpha) \\
Z'_{\epsilon-}(\alpha) &= \mathrm{i}\exp\left(-\frac{\mathrm{i}}{4}\epsilon\alpha\right)\left[\exp\left(\frac{\epsilon\pi}{4}\right)\chi'_\epsilon V'_\epsilon(\alpha) + \exp\left(-\frac{\epsilon\pi}{4}\right)V'_\epsilon(\alpha)\chi'_\epsilon\right]
\end{aligned} \tag{22}$$

where $V_\epsilon(\beta)$, $V'_\epsilon(\alpha)$, χ_ϵ and χ'_ϵ are defined through

$$\begin{aligned}
V_\epsilon(\beta) &\equiv \exp(\mathrm{i}\phi_\epsilon(\beta)) = \exp\left(-\frac{\mathrm{i}}{4}\epsilon\beta\right)\rho_\epsilon : \exp(\mathrm{i}\phi_\epsilon(\beta)) : \\
\langle u|\chi_\epsilon|v\rangle &\equiv \eta_\epsilon^{-1}\langle u|\int_c \frac{\mathrm{d}\gamma}{2\pi}\bar{V}_\epsilon(\gamma)|v\rangle \\
\bar{V}_\epsilon(\gamma) &\equiv \exp(-\mathrm{i}\bar{\phi}_\epsilon(\gamma)) = \exp(-\mathrm{i}\epsilon\gamma)\bar{\rho}_\epsilon : \exp(-\mathrm{i}\bar{\phi}_\epsilon(\gamma)) : \\
\bar{\phi}_\epsilon(\gamma) &\equiv \phi_\epsilon\left(\gamma + \frac{\mathrm{i}\pi}{2}\right) + \phi_\epsilon\left(\gamma - \frac{\mathrm{i}\pi}{2}\right) \\
V'_\epsilon(\alpha) &\equiv \exp(\mathrm{i}\phi'_\epsilon(\alpha)) = \exp\left(-\frac{\mathrm{i}}{4}\epsilon\alpha\right)\rho'_\epsilon : \exp(\mathrm{i}\phi'_\epsilon(\alpha)) : \\
\langle u|\chi'_\epsilon|v\rangle &\equiv \eta_\epsilon'^{-1}\langle u|\int_c \frac{\mathrm{d}\delta}{2\pi}\bar{V}'_\epsilon(\delta)|v\rangle
\end{aligned}$$

$$\begin{aligned}\bar{V}'_\epsilon(\delta) &\equiv \exp(-\mathrm{i}\bar{\phi}'_\epsilon(\delta)) = \exp(-\mathrm{i}\epsilon\delta)\bar{\rho}'_\epsilon : \exp(-\mathrm{i}\bar{\phi}'_\epsilon(\delta)) : \\ \bar{\phi}'_\epsilon(\delta) &\equiv \phi'_\epsilon\left(\delta + \frac{\mathrm{i}\pi}{2}\right) + \phi'_\epsilon\left(\delta - \frac{\mathrm{i}\pi}{2}\right).\end{aligned} \tag{23}$$

$Z_\epsilon(\beta)$ and $Z'_\epsilon(\alpha)$ satisfy the ZF algebra with the 'regularized' S-matrix respectively

$$\begin{aligned}(S^{cd}_{\epsilon ab}(\beta)) = (-S'^{cd}_{\epsilon ab}(-\beta)) &= \begin{pmatrix} 1 & & & \\ & \frac{\sinh\frac{1}{2}\epsilon\beta}{\sinh\frac{1}{2}\epsilon(\mathrm{i}\pi-\beta)} & \frac{-\sinh\frac{1}{2}\epsilon\pi}{\sinh\frac{1}{2}\epsilon(\mathrm{i}\pi-\beta)} & \\ & \frac{-\sinh\frac{1}{2}\epsilon\pi}{\sinh\frac{1}{2}\epsilon(\mathrm{i}\pi-\beta)} & \frac{\sinh\frac{1}{2}\epsilon\beta}{\sinh\frac{1}{2}\epsilon(\mathrm{i}\pi-\beta)} & \\ & & & 1\end{pmatrix} S_\epsilon(\beta) \\ S_\epsilon(\beta) &= \exp\left(-\frac{\mathrm{i}}{2}\epsilon\beta\right)\frac{g_\epsilon(-\beta)}{g_\epsilon(\beta)} \\ g_\epsilon(\beta) &= (1-q)^{\frac{1}{2}}\frac{\Gamma_q\left(\frac{\mathrm{i}\pi-\beta}{2\mathrm{i}\pi}\right)}{\Gamma_q\left(\frac{-\beta}{2\mathrm{i}\pi}\right)} \qquad (q = \exp(-2\epsilon\pi))\end{aligned} \tag{24}$$

where $\Gamma_q(x)$ is the quantum Γ-function,

$$\Gamma_q(x) \equiv (1-q)^{1-x}\prod_{k=1}^{+\infty}\frac{1-q^k}{1-q^{x+k-1}}. \tag{25}$$

In the limit $q \to 1$, $\Gamma_q(x)$ becomes the usual $\Gamma(x)$. So, it is easy to check that the q-bosonic field $\phi_\epsilon(\beta)$ and $\phi'_\epsilon(\alpha)$ satisfies

$$\begin{aligned}&[\phi_\epsilon(\beta_1), \phi_\epsilon(\beta_2)] = \ln S_\epsilon(\beta_2 - \beta_1) \\ &\langle\phi_\epsilon(\beta_1)\phi_\epsilon(\beta_2)\rangle = -\ln g_\epsilon(\beta_2 - \beta_1) \\ &[\phi'_\epsilon(\alpha_1), \phi'_\epsilon(\alpha_2)] = \ln S'_\epsilon(\alpha_2 - \alpha_1) \\ &\langle\phi'_\epsilon(\alpha_1)\phi'_\epsilon(\alpha_2)\rangle = -\ln g'_\epsilon(\alpha_2 - \alpha_1).\end{aligned} \tag{26}$$

3. *SU*(2)-invariant Thirring model with boundary

In this section, we briefly review our early work [21] about the boundary reflection matrix and boundary operators in the $SU(2)$-invariant Thirring model, and discuss the boundary bound states. We also discuss the $SU(2)ITM$ with a boundary in Lukyanov's π_z representation. The q-oscillator expressions for the $Z_a(\beta)$ and $Z'_a(\alpha)$ operators will be the same as those in the bulk theory because the commutation relations satisfied by local and asymptotic operators are unchanged. It is interesting to discuss the situation in which the direction of time evolution is parallel to the boundary. The Hilbert space of physics states H_B consists of wavefunctions defined on the semi-infinite line, $-\infty < x \leqslant 0$ at $t =$ constant while $t \in (-\infty, +\infty)$. The boundary ground states of Hilbert space will be given by the action of the boundary operators on the Fock vacuum†,

$$|0\rangle_B = \mathrm{e}^{\psi-}|0\rangle \qquad {}_B\langle 0| = \langle 0|\mathrm{e}^{\psi+} \tag{27}$$

which are the initial and final time boundary ground states respectively. The boundary ground states (27) are required to satisfy the following reflection equation of asymptotic operators $Z_a(\beta)$,

$$\begin{aligned}&Z_a(\beta)|0\rangle_B = R^b_a(\beta)Z_b(-\beta)|0\rangle_B \\ &{}_B\langle 0|Z_a(\mathrm{i}\pi + \beta) = {}_B\langle 0|Z_b(\mathrm{i}\pi - \beta)R^d_c(-\beta)C_{ca}C_{db}\end{aligned} \tag{28}$$

† For consistency with [18, 19], we will adopt $|0\rangle_B$ to express the boundary state instead of $|B\rangle$ as used in [21].

where the boundary reflection matrix must satisfy the boundary Yang–Baxter equation (BYBE) [18, 22, 23]

$$R_{a_2}^{c_2}(\beta_2)S_{a_1c_2}^{c_1d_2}(\beta_1+\beta_2)R_{c_1}^{d_1}(\beta_1)S_{d_2d_1}^{b_2b_1}(\beta_1-\beta_2)=S_{a_1a_2}^{c_1c_2}(\beta_1-\beta_2)R_{c_1}^{d_1}(\beta_1)S_{c_2d_1}^{d_2b_1}(\beta_1+\beta_2)R_{d_2}^{b_2}(\beta_2) \tag{29}$$

the boundary unitary condition

$$R_a^c(\beta)R_c^b(-\beta)=\delta_a^b \tag{30}$$

and boundary crossing-symmetry condition

$$R_a^b\left(-\frac{\mathrm{i}\pi}{2}-\beta\right)C_{ae}=S_{cf}^{eb}(2\beta)R_d^c\left(-\frac{\mathrm{i}\pi}{2}+\beta\right)C_{df}. \tag{31}$$

For $SU(2)ITM$, we have the following reflection matrix (in the case of the diagonal matrix):

$$(R_a^b(\beta))=\begin{pmatrix}1 & \\ & \frac{\mu-\frac{\mathrm{i}\pi}{2}-\beta}{\mu-\frac{\mathrm{i}\pi}{2}+\beta}\end{pmatrix}R(\beta) \tag{32}$$

where μ is the boundary parameter, $R(\beta)$ is a scalar function. In order to express the boundary state with q-oscillators, it is necessary to obtain the 'ultraviolet regularized' R-matrix from (24) and (29)–(31)

$$(R_{\epsilon a}^b(\beta))=\begin{pmatrix}1 & \\ & \frac{\sinh\frac{\mathrm{i}}{2}\epsilon\left(\mu-\frac{\mathrm{i}\pi}{2}-\beta\right)}{\sinh\frac{\mathrm{i}}{2}\epsilon\left(\mu-\frac{\mathrm{i}\pi}{2}+\beta\right)}\end{pmatrix}R_\epsilon(\beta) \tag{33}$$

where the scalar function $R_\epsilon(\beta)$ can be Riemann–Hilbert decomposed;

$$R_\epsilon(\beta)=\frac{f_\epsilon(-\beta,\mu)}{f_\epsilon(\beta,\mu)}$$

$$f_\epsilon(\beta,\mu)=\frac{\Gamma_{q^2}\left(\frac{(\mathrm{i}\pi/2)-\beta}{2\mathrm{i}\pi}\right)\Gamma_q\left(\frac{(3\mathrm{i}\pi/2)-\beta+\mu}{2\mathrm{i}\pi}\right)}{\Gamma_{q^2}\left(\frac{-\beta}{2\mathrm{i}\pi}\right)\Gamma_q\left(\frac{(\mathrm{i}\pi/2)-\beta+\mu}{2\mathrm{i}\pi}\right)} \tag{34}$$

$f_\epsilon(\beta,\mu)$ is required to be an analytical function in the lower half plane ($\mathrm{Im}\,\beta\leqslant 0$). The regularized form of equation (28) reads

$$\begin{aligned}&Z_{\epsilon a}(\beta)|0\rangle_B=R_{\epsilon a}^b(\beta)Z_{\epsilon b}(-\beta)|0\rangle_B\\ &{}_B\langle 0|Z_{\epsilon a}(\mathrm{i}\pi+\beta)={}_B\langle 0|Z_{\epsilon b}(\mathrm{i}\pi-\beta)R_{\epsilon c}^d(-\beta)C_{ca}C_{db}.\end{aligned} \tag{35}$$

Jimbo *et al* [19] gave a good way to express the boundary states with q-oscillators. Here we will express $|0\rangle_B$ in a similar way. We assumed that the initial time boundary ground state $|0\rangle_B$ can be expressed as

$$\begin{aligned}|0\rangle_B&=\mathrm{e}^{\psi_-}|0\rangle\\ \psi_-&=\sum_{n=1}^{+\infty}\alpha_n\frac{n\exp\left(-\frac{n\epsilon\pi}{2}\right)}{2\sinh\frac{n\epsilon\pi}{2}\sinh n\epsilon\pi}a_{-n}^2+\sum_{n=1}^{+\infty}\lambda_n\frac{n}{\sinh\frac{n\epsilon\pi}{2}}a_{-n}\end{aligned} \tag{36}$$

where α_n and λ_n are the parameters to be determined. Following the fundamental commutation relation of the q-oscillators (21), we have the following Bogoliubov transform:

$$\begin{aligned}\mathrm{e}^{-\psi_-}a_n\mathrm{e}^{\psi_-}&=a_n+\alpha_na_{-n}+\lambda_n\mathrm{e}^{\frac{n\epsilon\pi}{2}}\sinh n\epsilon\pi\\ \mathrm{e}^{-\psi_-}a_{-n}e^{\psi_-}&=a_{-n}\qquad n>0.\end{aligned} \tag{37}$$

Substituting expression (36) into (35) and using (37), we get

$$\begin{aligned}\alpha_n&=-1\\ \lambda_n&=-\frac{q^{((\mathrm{i}\pi+\mu)/2\mathrm{i}\pi)n}}{n\left(1+q^{\frac{n}{2}}\right)}+\theta_n\frac{q^{\frac{n}{2}}-q^{\frac{n}{4}}}{n\left(1+q^{\frac{n}{2}}\right)}\end{aligned} \tag{38}$$

where

$$\theta_n = \begin{cases} 0, & n : \text{odd} \\ 1, & n : \text{even.} \end{cases} \tag{39}$$

In the similar way, we may express the final time boundary ground state:

$$\begin{aligned} {}_B\langle 0| &= \langle 0|\mathrm{e}^{\psi_+} \\ \psi_+ &= \sum_{n=1}^{+\infty} \sigma_n \frac{n \exp\left(-\frac{n\epsilon\pi}{2}\right)}{2\sinh\frac{n\epsilon\pi}{2}\sinh n\epsilon\pi} a_n^2 + \sum_{n=1}^{+\infty} \rho_n \frac{n}{\sinh\frac{n\epsilon\pi}{2}} a_n \end{aligned} \tag{40}$$

where

$$\begin{aligned} \sigma_n &= -q^n \\ \rho_n &= -\frac{q^{((\mathrm{i}\pi+\mu)/2\mathrm{i}\pi)n}}{n\left(1+q^{\frac{n}{2}}\right)} + \theta_n \frac{q^{\frac{3}{4}n} - q^n}{n\left(1+q^{\frac{n}{2}}\right)}. \end{aligned} \tag{41}$$

Utilizing the operators $Z'_a(\alpha)$, we can construct an operator $T(\alpha)$:

$$T(\alpha) = \frac{1}{\mathrm{i}} C_{ab} Z'_a(\alpha + \mathrm{i}\pi) R'^c_b(\alpha) Z'_c(-\alpha). \tag{42}$$

Here, $(R'^b_a(\alpha))$ is a 2×2 matrix:

$$(R'^b_{\epsilon a}(\alpha)) = \begin{pmatrix} 1 & \\ & \frac{\sin(\mathrm{i}/2)\epsilon(\mu+\alpha)}{\sinh(\mathrm{i}/2)\epsilon(\mu-\alpha)} \end{pmatrix} R'_\epsilon(\alpha) \tag{43}$$

$$R'_\epsilon(\alpha) = \frac{\Gamma_{q^2}\left(\frac{(3\mathrm{i}\pi/2)-\alpha}{2\mathrm{i}\pi}\right)\Gamma_{q^2}\left(\frac{2\mathrm{i}\pi+\alpha}{2\mathrm{i}\pi}\right)\Gamma_q\left(\frac{\mu+\alpha+\mathrm{i}\pi}{2\mathrm{i}\pi}\right)\Gamma_q\left(\frac{\mu-\alpha+2\mathrm{i}\pi}{2\mathrm{i}\pi}\right)}{\Gamma_{q^2}\left(\frac{(3\mathrm{i}\pi/2)+\alpha}{2\mathrm{i}\pi}\right)\Gamma_{q^2}\left(\frac{2\mathrm{i}\pi-\alpha}{2\mathrm{i}\pi}\right)\Gamma_q\left(\frac{\mu-\alpha+\mathrm{i}\pi}{2\mathrm{i}\pi}\right)\Gamma_q\left(\frac{\mu+\alpha+2\mathrm{i}\pi}{2\mathrm{i}\pi}\right)}. \tag{44}$$

$R'(\alpha)$ and $S'(\alpha)$ also satisfy equations (29)–(31). We can prove that the $|0\rangle_B$ expressed in (36) is the eigenstate of $T(\alpha)$:

$$T(\alpha)|0\rangle_B = |0\rangle_B. \tag{45}$$

Using (2), we get the following equation satisfied by $Z'_a(\alpha)$,

$$\begin{aligned} Z'_a(\alpha)|0\rangle_B &= R'^b_a(\alpha) Z'_b(-\alpha)|0\rangle_B \\ {}_B\langle 0|Z'_c(\alpha+\mathrm{i}\pi)C_{ac} &= {}_B\langle 0|Z'_d(-\alpha+\mathrm{i}\pi)C_{bd}R'^a_b(-\alpha). \end{aligned} \tag{46}$$

Now we discuss the singularity structure of the boundary scattering amplitude $R^b_a(\beta)$. According to the discussion of the boundary SGM in [18] and boundary XXZ model in [19], poles due to boundary bound states should occur in the 'physics strip' $0 \leqslant \operatorname{Im}\beta < \frac{\pi}{2}$. To ensure that the boundary bound states are physics states, poles are also required in the 'well defined regions' which are discussed in detail in [19]. From (33), (34), it is known that $R^b_a(\beta)$ has a pole at $\beta = -\mu + \frac{\mathrm{i}\pi}{2}$ which may be in a physical strip if we select an appropriate μ, $0 < \operatorname{Im}\mu \leqslant \frac{\pi}{2}$. Note here that the calculations of boundary $SU(2)ITM$ discussed in this paper is similar to that of Jimbo's boundary XXZ model [19] in the limit of $q^{\frac{1}{4}} \to -1$, if we let

$$q^{\frac{1}{4}} = -\mathrm{e}^{-\frac{\pi\epsilon}{2}} \qquad r = -\mathrm{e}^{\mathrm{i}\mu\epsilon} \qquad \xi^2 = \mathrm{e}^{\mathrm{i}\beta\epsilon}.$$

This implies that the pole $\beta = -\mu + \frac{\mathrm{i}\pi}{2}$ is just in a well-defined region†. This pole corresponds to single-particle-boundary bound state. For example, state $Z_{-\epsilon}(\beta)|0\rangle_B$ has a

† This region corresponds to region (i) in the boundary XXZ model in [19].

series of simple poles at $\beta = 0, \frac{i\pi}{2}, -\mu - \frac{i\pi}{2}, -\mu + \frac{i\pi}{2}, \mu + \frac{i\pi}{2}$. After a straightforward calculation, we find only at $\beta = -\mu + \frac{i\pi}{2}$ the residue is not zero

$$\mathrm{Res}_{(\beta=-\mu+(i\pi/2))} Z_{-\epsilon}(\beta)|0\rangle_B = i\eta^{-1}\rho\bar{\rho}(2\pi i)^2 \frac{\Gamma\left(\frac{i\pi-\mu}{2i\pi}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{(i\pi/2)-\mu}{2i\pi}\right)} \times e^{-i\phi^-\left(2\mu+\frac{i\pi}{2}\right)-i\phi^-\left(-\frac{i\pi}{2}-\mu\right)} e^{\psi_-}|-\sqrt{\tfrac{1}{2}}\rangle \tag{47}$$

where $|-\sqrt{\frac{1}{2}}\rangle = \exp(-i\sqrt{\frac{1}{2}}Q)|0\rangle$. We can follow the discussion in [19] and interpret

$$|-\sqrt{\tfrac{1}{2}}\rangle_B = \exp\left[\psi_- - i\phi^-\left(2\mu + \frac{i\pi}{2}\right) - i\phi^-\left(-\mu - \frac{i\pi}{2}\right)\right]|-\sqrt{\tfrac{1}{2}}\rangle$$

as the boundary bound states. It is another boundary ground state with higher energy than that of $|0\rangle_B$. $|0\rangle_B$ and $|-\sqrt{\frac{1}{2}}\rangle_B$ are two boundary ground states corresponding to different sectors of the Fock space.

4. *N*-point correlation function

In this section, we calculate the N-point function of regularized local operators $Z'_\epsilon(\alpha)$, from which we get the N-point correlation function of boundary $SU(2)ITM$ when we let $q^{\frac{1}{4}} \to -1$.

For later use, let us define the coherent state in the Fock spaces (in the following, we shall suppress the label 0 for vacuum state $|0\rangle_0$ before $q^{\frac{1}{4}} \to -1$ if there is no fear of confusion).

$$|\xi'\rangle = \prod_{n=1}^{+\infty} |\xi'_n\rangle$$

$$|\xi'_n\rangle = \exp\left(e^{\frac{n\epsilon\pi}{2}} \frac{n\xi'_n}{\sinh\frac{n\epsilon\pi}{2}\sinh n\epsilon\pi} a'_-\right)|0\rangle \tag{48}$$

$$\langle\bar{\xi}'| = \prod_{n=1}^{+\infty} \langle\bar{\xi}'_n|$$

$$\langle\bar{\xi}'_n| = \langle 0|\exp\left(e^{\frac{n\epsilon\pi}{2}} \frac{n\bar{\xi}'_n}{\sinh\frac{n\epsilon\pi}{2}\sinh n\epsilon\pi} a'_+\right) \tag{49}$$

where ξ'_n and $\bar{\xi}'_n$ are complex conjugate parameters. They satisfy

$$a'_n|\xi'_m\rangle = \xi'_m|\xi'_m\rangle\delta_{nm}$$
$$\langle\bar{\xi}'_m|a'_{-n} = \langle\bar{\xi}'_m|\bar{\xi}'_m\delta_{nm} \tag{50}$$

and the completeness relation

$$\mathrm{i}d = \int \prod_{n=1}^{+\infty} \frac{n\exp\left(\frac{n\epsilon\pi}{2}\right) d\xi'_n\, d\bar{\xi}'_n}{\pi\sinh\frac{n\epsilon\pi}{2}\sinh n\epsilon\pi} \exp\left(\sum_{n=1}^{+\infty} \frac{n\exp\left(\frac{n\epsilon\pi}{2}\right)}{\sinh\frac{n\epsilon\pi}{2}\sinh n\epsilon\pi}|\xi'_n|^2\right)|\xi'_n\rangle\langle\bar{\xi}'|. \tag{51}$$

N-point correlation functions are the vacuum expectation values of products of local operators. They can be defined as the following function with N even,

$$F_N = \frac{{}_B\langle 0|Z'_{\epsilon a_1}(\alpha_1)Z'_{\epsilon a_2}(\alpha_2)\cdots Z'_{\epsilon a_N}(\alpha_N)|0\rangle_B}{{}_B\langle 0|0\rangle_B} \tag{52}$$

where a_j is an isotopic number taking values $+1$ or -1. Since the total spin is conserved, we require $\sum_{j=1}^{N} a_j = 0$. We denote by A the index set

$$A = \{j | 1 \leqslant j \leqslant N, a_j = -1\}. \tag{53}$$

Because (35) is homogeneous equation, we can redefine the boundary state as follows,

$$\begin{aligned} |0\rangle_B &= (1-q^2)^{\frac{1}{4}} (q^2; q^2)_\infty^{\frac{1}{2}} \mathrm{e}^{\psi_-} |0\rangle \\ {}_B\langle 0| &= \langle 0| \mathrm{e}^{\psi_+} (1-q^2)^{\frac{1}{4}} (q^2; q^2)_\infty^{\frac{1}{2}} \end{aligned} \tag{54}$$

where

$$\begin{aligned} (z; q)_\infty &= \prod_{k=0}^{\infty} (1 - zq^k) \\ (z; p, q)_\infty &= \prod_{j,k \geqslant 0} (1 - zp^j q^k). \end{aligned} \tag{55}$$

Then,

$${}_B\langle 0|0\rangle_B = (1-q^2)^{\frac{1}{2}} (q^2; q^2)_\infty \langle 0| \mathrm{e}^{\psi_+} \mathrm{e}^{\psi_-} |0\rangle. \tag{56}$$

Inserting between e^{ψ_+} and e^{ψ_-} the completeness relation of the coherent states (51) and using (50), we have

$${}_B\langle 0|0\rangle_B = (1-q^2)^{\frac{1}{2}} (q^2; q^2)_\infty \prod_{n=1}^{+\infty} Q_n \tag{57}$$

$$\begin{aligned} Q_n &= (1 - \sigma_n \alpha_n)^{-\frac{1}{2}} \exp\left[\frac{n(1 + \mathrm{e}^{n\epsilon\pi})(2\rho_n \lambda_n + \rho_n^2 \alpha_n + \lambda_n^2 \sigma_n)}{2(1 - \sigma_n \alpha_n)} \right] \\ &= (1 - q^n)^{-\frac{1}{2}} \exp\left[\frac{q^{((\mathrm{i}\pi + 2\mu)/2\mathrm{i}\pi)n}}{2n(1 + q^{\frac{n}{2}})} - \theta_n \frac{q^{\frac{n}{2}} (1 - q^{\frac{n}{4}})^2}{n(1 - q^n)} - \theta_n \frac{q^{(n\mu/2\mathrm{i}\pi)} (q^{\frac{n}{2}} - q^{\frac{n}{4}})}{n(1 + q^{\frac{n}{2}})} \right] \end{aligned} \tag{58}$$

where α_n, λ_n, ρ_n, σ_n are defined in (38), (41). Thus, we have

$$\begin{aligned} {}_B\langle 0|0\rangle_B &= (1-q^2)^{\frac{1}{2}} (q^2; q^2)_\infty \frac{\left(q^{((2\mathrm{i}\pi + 2\mu)/2\mathrm{i}\pi)}; q^2\right)_\infty}{\left(q^{((\mathrm{i}\pi + 2\mu)/2\mathrm{i}\pi)}; q^2\right)_\infty (q^{3/2}; q^2)_\infty} \\ &= \frac{\Gamma_{q^2}\left(\frac{(\mathrm{i}\pi/2) + \mu}{2\mathrm{i}\pi}\right) \Gamma_{q^2}\left(\frac{3}{4}\right)}{\Gamma_{q^2}\left(\frac{\mathrm{i}\pi + \mu}{2\mathrm{i}\pi}\right)}. \end{aligned} \tag{59}$$

From (57) to (59), we use the formula

$$\begin{aligned} \ln \Gamma_q(x) &= (1-x)\ln(1-q) + \ln(q; q)_\infty - \ln(q^x; q)_\infty \\ &= (1-x)\ln(1-q) - \sum_{n=1}^{+\infty} \frac{q^n + q^{xn}}{n(1-q^n)}. \end{aligned} \tag{60}$$

In order to evaluate the expectation value (52), we substitute the bosonization formulae (22) into (52) and normal-order the product of vertex operators, and have

$$F_N = \mathrm{i}^{\frac{N}{2}} \rho_\epsilon'^N \bar{\rho}_\epsilon'^{\frac{N}{2}} \eta_\epsilon'^{-\frac{N}{2}} \prod_{j=1}^{N-1} \prod_{l>j}^{N} \mathrm{e}^{-\frac{\mathrm{i}\epsilon\alpha_j}{2}} g_\epsilon'(\alpha_l - \alpha_j)$$
$$\times \prod_{a\in A} \left\{ \int_{C_a} \frac{\mathrm{d}\delta_a}{2\pi} \left[\mathrm{e}^{-\frac{\mathrm{i}\epsilon(\alpha_a + (\mathrm{i}\pi/2))}{2}} w_\epsilon'(\alpha_a - \delta_a) + \mathrm{e}^{\frac{\mathrm{i}\epsilon(\alpha_a + (\mathrm{i}\pi/2) - 2\delta_a)}{2}} w_\epsilon'(\delta_a - \alpha_a) \right] \right.$$
$$\left. \times \prod_{k>a} \mathrm{e}^{\mathrm{i}\epsilon\delta_a} w_\epsilon'(\alpha_k - \delta a) \prod_{k<a} \mathrm{e}^{\mathrm{i}\epsilon\alpha_k} w_\epsilon'(\delta_a - \alpha_k) \prod_{b>a} \mathrm{e}^{-2\mathrm{i}\epsilon\delta_a} \bar{g}_\epsilon'(\delta_b - \delta_a) \right\}$$
$$\times \frac{I(\{\alpha_j\}, \{\delta_a\})}{{}_B\langle 0|0\rangle_B} \tag{61}$$

where

$$w_\epsilon'(\alpha) = \left[g_\epsilon'\left(\alpha + \frac{\mathrm{i}\pi}{2}\right) g_\epsilon'\left(\alpha - \frac{\mathrm{i}\pi}{2}\right)\right]^{-1} = \left[1 - \mathrm{e}^{\mathrm{i}\epsilon\left(\frac{\mathrm{i}\pi}{2} - \alpha\right)}\right]^{-1} \tag{62}$$

$$\bar{g}_\epsilon'(\alpha) = \left[w_\epsilon'\left(\alpha + \frac{\mathrm{i}\pi}{2}\right) w_\epsilon'\left(\alpha - \frac{\mathrm{i}\pi}{2}\right)\right]^{-1} = (1 - \mathrm{e}^{-\mathrm{i}\epsilon\alpha})(1 - \mathrm{e}^{\mathrm{i}\epsilon(\mathrm{i}\pi - \alpha)}). \tag{63}$$

Each contour C_a is the same one defined in (12) as C and

$$I(\{\alpha_j\}, \{\delta_a\}) = {}_B\langle 0| \exp\left(\sum_{n=1}^{\infty} a_{-n} X_n\right) \exp\left(-\sum_{n=1}^{\infty} a_n Y_n\right) |0\rangle_B \tag{64}$$

where

$$X_n = \sum_{j=1}^{N} \frac{\mathrm{e}^{-\mathrm{i}\epsilon\alpha_j n}}{\sinh(n\epsilon\pi)} - \sum_{a\in A} \frac{\mathrm{e}^{-\mathrm{i}\epsilon\delta_a n}}{\sinh\left(\frac{n\epsilon\pi}{2}\right)}$$
$$Y_n = \sum_{j=1}^{N} \frac{\mathrm{e}^{\mathrm{i}\epsilon\alpha_j n}}{\sinh(n\epsilon\pi)} - \sum_{a\in A} \frac{\mathrm{e}^{\mathrm{i}\epsilon\delta_a n}}{\sinh\left(\frac{n\epsilon\pi}{2}\right)}. \tag{65}$$

In a similar way to that used in calculating (57), we have

$$I(\{\alpha_j\}, \{\delta_a\}) = {}_B\langle 0|0\rangle_B \prod_{n=1}^{\infty} \exp\left\{ \frac{n(1 + \mathrm{e}^{n\epsilon\pi})}{2(1 - \sigma_n\alpha_n)} \left[\frac{\sigma_n(1 - \mathrm{e}^{-n\epsilon\pi})^2}{4n^2} X_n^2 + \frac{\alpha_n(1 - \mathrm{e}^{-n\epsilon\pi})^2}{4n^2} Y_n^2 \right.\right.$$
$$- \frac{\alpha_n\sigma_n(1 - \mathrm{e}^{-n\epsilon\pi})^2}{2n^2} X_n Y_n + \frac{1 - \mathrm{e}^{-n\epsilon\pi}}{n} (\rho_n + \sigma_n\lambda_n) X_n$$
$$\left.\left. - \frac{1 - \mathrm{e}^{-n\epsilon\pi}}{n} (\lambda_n + \alpha_n\rho_n) Y_n \right]\right\}. \tag{66}$$

So

$$\frac{I(\{\alpha_j\}, \{\delta_a\})}{{}_B\langle 0|0\rangle_B} = \prod_{k<j} \left[\left(q^{\frac{3\mathrm{i}\pi - \alpha_j - \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \left(q^{\frac{\mathrm{i}\pi + \alpha_j + \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \left(q^{\frac{3\mathrm{i}\pi + \alpha_j - \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \right.$$
$$\left. \times \left(q^{\frac{3\mathrm{i}\pi - \alpha_j + \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \right] \left[\left(q^{\frac{4\mathrm{i}\pi - \alpha_j - \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \left(q^{\frac{2\mathrm{i}\pi + \alpha_j + \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \right.$$
$$\left. \times \left(q^{\frac{4\mathrm{i}\pi + \alpha_j - \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \left(q^{\frac{4\mathrm{i}\pi - \alpha_j + \alpha_k}{2\mathrm{i}\pi}}; q, q\right)_\infty \right]^{-1}$$
$$\times \prod_{b>a} \left[\left(q^{\frac{2\mathrm{i}\pi - \delta_a - \delta_b}{2\mathrm{i}\pi}}; q^{\frac{1}{2}}\right)_\infty \left(q^{\frac{\delta_a + \delta_b}{2\mathrm{i}\pi}}; q^{\frac{1}{2}}\right)_\infty \left(q^{\frac{2\mathrm{i}\pi + \delta_a - \delta_b}{2\mathrm{i}\pi}}; q^{\frac{1}{2}}\right)_\infty \left(q^{\frac{2\mathrm{i}\pi - \delta_a + \delta_b}{2\mathrm{i}\pi}}; q^{\frac{1}{2}}\right)_\infty \right]$$

$$
\begin{aligned}
&\times \prod_{j=1}^{N}\prod_{a\in A}\Big[\Big(q^{\frac{(5i\pi/2)-\alpha_j-\delta_a}{2i\pi}};q\Big)_\infty \Big(q^{\frac{(i\pi/2)+\alpha_j+\delta_a}{2i\pi}};q\Big)_\infty \\
&\times \Big(q^{\frac{(5i\pi/2)+\alpha_j-\delta_a}{2i\pi}};q\Big)_\infty \Big(q^{\frac{(5i\pi/2)-\alpha_j+\delta_a}{2i\pi}};q\Big)_\infty\Big]^{-1} \\
&\times \prod_{j=1}^{N}\Big[\Big(q^{\frac{5i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{7i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{3i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \\
&\times \Big(q^{\frac{5i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{i\pi+\mu-\alpha_j}{2i\pi}};q\Big)_\infty\Big]\Big[\Big(q^{\frac{6i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \\
&\times \Big(q^{\frac{3i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{4i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{6i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \\
&\times \Big(q^{\frac{2i\pi+\mu-\alpha_j}{2i\pi}};q\Big)_\infty\Big]^{-1}\left[\frac{\Big(q^{\frac{3}{2}};q,q\Big)_\infty}{\Big(q^{\frac{3}{2}};q,q\Big)_\infty}\right]^{N} \\
&\times \prod_{a\in A}\left[\frac{\Big(q^{\frac{\delta_a}{i\pi}};q\Big)_\infty \Big(q^{\frac{i\pi-\delta_a}{i\pi}};q\Big)_\infty}{1-q^{\frac{(i\pi/2)+\mu-\delta_a}{2i\pi}}}\right]\Big(q;q^{\frac{1}{2}}\Big)_\infty^{\frac{N}{2}}. \qquad (67)
\end{aligned}
$$

Now, we discuss the limit result when $q^{\frac{1}{4}} \to -1$. Let us define

$$
A(x,;a,b) = \exp\left(-\int_0^\infty \frac{dt}{t}\frac{\sinh^2(xt)\,e^{-\frac{b}{2}t}}{\sinh at\cosh\frac{t}{2}}\right). \qquad (68)
$$

In the limit $q^{\frac{1}{4}} \to -1$, we have

$$
\begin{aligned}
&\Gamma_q(x) \mapsto \Gamma(x) \\
&\Gamma_{q^2}(x) \mapsto \Gamma(x) \\
&\frac{\Big(q^{\frac{i\pi+\alpha_j+\alpha_k}{2i\pi}};q,q\Big)_\infty \Big(q^{\frac{3i\pi-\alpha_j-\alpha_k}{2i\pi}};q,q\Big)_\infty}{\Big(q^{\frac{2i\pi+\alpha_j+\alpha_k}{2i\pi}};q,q\Big)_\infty \Big(q^{\frac{4i\pi-\alpha_j-\alpha_k}{2i\pi}};q,q\Big)_\infty} \mapsto (1-q)^{\frac{1}{4}}\frac{(q;q)_\infty A\left(\frac{\alpha_j+\alpha_k-i\pi}{2i\pi};1,1\right)_\infty}{\Gamma\left(\frac{3}{4}\right)A\left(\frac{1}{4};1,1\right)_\infty} \\
&\frac{\Big(q^{\frac{3i\pi+\alpha_j-\alpha_k}{2i\pi}};q,q\Big)_\infty \Big(q^{\frac{3i\pi-\alpha_j+\alpha_k}{2i\pi}};q,q\Big)_\infty}{\Big(q^{\frac{4i\pi+\alpha_j-\alpha_k}{2i\pi}};q,q\Big)_\infty \Big(q^{\frac{4i\pi-\alpha_j+\alpha_k}{2i\pi}};q,q\Big)_\infty} \mapsto (1-q)^{-\frac{1}{4}}\frac{(q;q)_\infty A\left(\frac{\alpha_j-\alpha_k}{2i\pi};1,3\right)_\infty}{\Gamma\left(\frac{5}{4}\right)A\left(\frac{1}{4};1,3\right)_\infty} \\
&\frac{\Big(q^{\frac{5i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{7i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{3i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{5i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty}{\Big(q^{\frac{6i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{3i\pi-2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{4i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty \Big(q^{\frac{6i\pi+2\alpha_j}{2i\pi}};q^2,q^2\Big)_\infty} \\
&\qquad \mapsto (1-q)^{\frac{1}{4}}\frac{(q^2;q^2)_\infty A\left(\frac{\alpha_j-(i\pi/2)}{2i\pi};2,3\right)}{\Gamma\left(\frac{7}{8}\right)A\left(\frac{1}{4};2,3\right)} \\
&\lim_{\epsilon\to 0}\rho'_\epsilon\bar{\rho}'^{\frac{1}{2}}_\epsilon\eta'^{-1/2}_\epsilon = \pi^{-\frac{1}{4}}(1-q)^{\frac{1}{4}}. \qquad (69)
\end{aligned}
$$

Considering (69), we have the following N-point function of $SU(2)ITM$:

$$\lim_{q^{\frac{1}{4}}\to -1} F_N = \mathrm{i}^{\frac{N}{2}} B_1 B_2 \prod_{l>j} \frac{\Gamma\left(\frac{2\mathrm{i}\pi-\alpha_l-\alpha_j}{2\mathrm{i}\pi}\right)}{\Gamma\left(\frac{\mathrm{i}\pi-\alpha_l-\alpha_j}{2\mathrm{i}\pi}\right)} A\left(\frac{\alpha_l+\alpha_j-\mathrm{i}\pi}{2\mathrm{i}\pi};1,1\right) A\left(\frac{\alpha_l-\alpha_j}{2\mathrm{i}\pi};1,3\right)$$
$$\times \prod_{j=1}^{N}\left[A\left(\frac{\alpha_j-\frac{\mathrm{i}\pi}{2}}{2\mathrm{i}\pi};2,3\right)\frac{\Gamma\left(\frac{2\mathrm{i}\pi+\mu-\alpha_j}{2\mathrm{i}\pi}\right)}{\Gamma\left(\frac{\mathrm{i}\pi+\mu-\alpha_j}{2\mathrm{i}\pi}\right)}\right]$$
$$\times \prod_{a\in A}\int_{C_a}\frac{\mathrm{d}\delta_a}{2\pi}\left\{\frac{\prod_{b>a}(\delta_a-\delta_b)(\delta_a-\delta_b+\mathrm{i}\pi)}{\prod_{k\geqslant a}\left(\delta_a-\alpha_k+\frac{\mathrm{i}\pi}{2}\right)\prod_{k\leqslant a}\left(\alpha_k-\delta_a+\frac{\mathrm{i}\pi}{2}\right)}\right.$$
$$\times\left[\frac{2\mathrm{i}\sin\frac{\delta_a}{\mathrm{i}}}{\pi\left(\frac{\mathrm{i}\pi}{2}+\mu-\delta_a\right)}\right]$$
$$\times\prod_{b>a}\left[\Gamma\left(\frac{2\mathrm{i}\pi-\delta_a-\delta_b}{\mathrm{i}\pi}\right)\Gamma\left(\frac{\delta_a+\delta_b}{\mathrm{i}\pi}\right)\right.$$
$$\left.\times\,\Gamma\left(\frac{2\mathrm{i}\pi+\delta_a-\delta_b}{\mathrm{i}\pi}\right)\Gamma\left(\frac{2\mathrm{i}\pi-\delta_a+\delta_b}{\mathrm{i}\pi}\right)\right]^{-1}$$
$$\times\prod_{j=1}^{N}\left[\Gamma\left(\frac{\frac{5\mathrm{i}\pi}{2}-\alpha_j+\delta_a}{2\mathrm{i}\pi}\right)\Gamma\left(\frac{\frac{\mathrm{i}\pi}{2}+\alpha_j+\delta_a}{2\mathrm{i}\pi}\right)\right.$$
$$\left.\left.\times\,\Gamma\left(\frac{\frac{5\mathrm{i}\pi}{2}+\alpha_j-\delta_a}{2\mathrm{i}\pi}\right)\Gamma\left(\frac{\frac{5\mathrm{i}\pi}{2}-\alpha_j+\delta_a}{2\mathrm{i}\pi}\right)\right]\right\} \tag{70}$$

where B_1 and B_2 are

$$B_1 = 2^{N^2-\frac{9}{8}N}\pi^{-\frac{3}{4}N^2+\frac{5}{4}N}$$
$$B_2 = \lim_{q^{\frac{1}{4}}\to -1}(1-q)^{-\frac{N}{8}}(q^2;q^2)_\infty^{-N}\left[\frac{(q^{\frac{3}{2}};q,q)_\infty}{(q^2;q,q)_\infty}\right]^N. \tag{71}$$

It is trivial to prove that B_2 is a finite constant. The contour C_a encircles the points

$$\alpha_k-\frac{\mathrm{i}\pi}{2} \qquad k\geqslant a$$
$$\alpha_j-\frac{5\mathrm{i}\pi}{2} \qquad -\alpha_j-\frac{\mathrm{i}\pi}{2} \qquad j=1,2,\ldots,N.$$

5. Discussion

In this article, we have only calculated the N-point correlators of the Jost function via regularized $Z'_{\epsilon a}(\alpha)$ operators. Using a similar approach, we can obtain the correlators including Z operators, which are more complicated. By taking the continuum limit, we obtain the N-point correlators of $SU(2)ITM$. According to Lukyanov's view [12–14], using (18) one can construct some bilinear local operators from Z' operators, which satisfy the mutual locality with Z operators. After taking the asymptotic limit, it is possible to obtain the form factors of local operators from correlators of Jost functions. Knowing the form factors one can reconstruct the correlation functions in the familiar manner [16, 24, 25]. This is the reason for adopting the word *correlation functional*, although strictly speaking we only obtained the *correlators*. The boundary case is the same as in the bulk, where the physical meaning of the N-point functions obtained by Lukyanov [13] utilizing the free bosonic expressions, and the relation between which and Smirnov's form-factors [16] are still interesting questions. We hope to discuss the reflect matrix in non-diagonal case. We have obtained q-oscillators realization of boundary state and form factor for sine–Gordon model [26]. They satisfy the same form factor axioms as that given by Jimbo *et al* [20]. We also hope to calculate the form factor for the other model, for example, $SU(n)$ Gross–Neven model, $SO(n) - \sigma$ model etc.

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